MATH 2230 B 03/03/2021.

Last week: Cauchy Integral Formula.

I. Differentiation Thm from CIF,

Let f be analytic inside & on a simple closed contour C, in counterclockwise direction. Then if z_0 is $\forall$ pt interior to C,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz, \quad n=0,1,2,3,\dots$$

Rk: CIF: $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)}\,dz$, so it coincides with the above formula.

Leibniz Rule:

For f defined on $\gamma \times G$, γ is of finite length path, G is open. Then for $\varphi(z) = \int_\gamma f(w,z)\,dw$, it is cts.

Moreover, if f is C^1 in z for $\gamma \times G$, then $\varphi'(z) = \int_\gamma \partial_z f(w,z)\,dw$.

Proof: Integrand of CIF: $\frac{f(w)}{w-z}$, $w \in C$,

& z is in the area interior to C,
$\underset{G}{\text{area}}$

$\frac{f(w)}{w-z}$ is C^1 in $\gamma \times G$,

then we can just apply Leibniz rule, we get $f'(z) = \frac{1}{2\pi i} \int_C \frac{\partial}{\partial z} \frac{f(w)}{w-z} \, dw$

$$= \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z)^2}.$$

Then just, proceed successively using the same idea for n. ☐

Proof of Leibniz Rule.

$\varphi(z) = \int_\gamma f(w, z) \, dw$,

First show continuity of φ:

$$|\varphi(z+\delta) - \varphi(z)| = \left| \int_\gamma f(w, z+\delta) - f(w,z) \right|$$

$$\leq \int_\gamma |f(w, z+\delta) - f(w,z)|$$

Because $\gamma \times B_r(z)$ is cpt, we know f is uniformly on it,
so for any ε, we can find

δ, s.t $|f(w, z+\delta) - f(w,z)| < \varepsilon$.

$|\varphi(z+\delta) - \varphi(z)| \leq \varepsilon L(\gamma)$.

Proceed to show differentiability given $f \in C^1$ w.r.t z,

$$\frac{\varphi(z+\varepsilon) - \varphi(z)}{\varepsilon} = \frac{\int_\gamma f(w, z+\varepsilon) - f(w,z)\, dw}{\varepsilon}$$

$$= \frac{\int_\gamma \int_0^1 \partial_t f(w, z+\varepsilon t)\, dt\, dw}{\varepsilon}$$

$$= \int_\gamma \int_0^1 \frac{\partial_z f(w, z+\varepsilon t)\varepsilon}{\varepsilon}\, dt\, dw$$

$$= \int_\gamma \int_0^1 \partial_z f(w, z+\varepsilon t)\, dt\, dw$$

$$\left| \frac{\varphi(z+\varepsilon) - \varphi(z)}{\varepsilon} - \underbrace{\int_\gamma \partial_z f(w, z)}_{\text{does not depend on } t} \right|$$

$$= \left| \int_\gamma \int_0^1 \partial_z f(w, z+\varepsilon t) - \partial_z f(w,z)\, dt\, dw \right|$$

$$\leq \int_\gamma \int_0^1 |\partial_z f(w, z+\varepsilon t) - \partial_z f(w,z)|\, dt\, dw.$$

Then again, we may take a closed ball centered at z & belongs to G, So $\gamma \times \overline{B_r(z)}$ is cpt, so $\partial_z f$ is uniformly cts on it. So $|\partial_z f(w, z+\varepsilon t) - \partial_z f(w, z)| < \varepsilon'$, $\forall \varepsilon'$, $\exists \varepsilon$.

$$\int_\gamma \int_0^1 |\partial_z f(w, z+\varepsilon t) - \partial_z f(w, z)| \, dt \, dw$$

$$\leq \int_\gamma \varepsilon \, dw = \varepsilon L(\gamma). \qquad \square$$

2. Consequence of the CIF.

1: If f is analytic at z_0, then its derivatives are also analytic.

Pf: $$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w-z)^{n+1}} \, dw,$$

where C is the bdy of some neighborhood of z_0 on which f is differentiable. $\square$

Some of the text book define analytic function to be C^1 on some open set. But ours only requires differentiability on open set.

If f is $C^1 \Rightarrow f$ is differentiable.

If f is differentiable on some open set, then $f'(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)^2}\,dw$, by Leibniz rule $f'(z)$ is cts, $\Rightarrow f \in C^1$ on some open set.

$f = u + iv$, u, v are harmonic,

$f' = u_x + iv_x = v_y - iu_y$ is also analytic

so cts. $\Rightarrow u, v \in C^1$.

Proceed successively, $u, v \in C^\infty$ (only in 2D case).

Actually, u, v belongs to real analytic functions.

3. Liouville Thm.

If f is entire & bdd, the $f \equiv c$.

Pf: f is analytic on any balls of arbitrary radius.

$$f'(z_0) = \frac{1}{2\pi i}\int_{B_R(z_0)} \frac{f(z)}{(z-z_0)^2}\,dz$$

$$|f'(z_0)| \leq \frac{1}{2\pi} \int_{B_R(z_0)} \frac{|f(z)|}{R^2} \, dz$$

$$\leq \frac{M}{R} \quad \text{for any } R.$$

So if take $R \to \infty$, $f'(z_0) = 0$ for any $z_0 \in \mathbb{C}$.

By the preceding thm: If $f' = 0$ as $\forall z \in G$, then $f \equiv c$.

$\Rightarrow f \equiv c$ on $\mathbb{C}$.

Fundamental Thm of calculus

Any polynomial has at lease one zero.

Proof: $f(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_0$, WLOG.

$$= z^n \left(1 + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right),$$

$$|f(z)| = \underbrace{|z|^n}_{\to \infty} \underbrace{\left| 1 + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right|}_{\to 1}$$

if $z \to \infty$

$|f(z)| \to \infty$

Assume f has no zeros, then we can define $g = \frac{1}{f}$, & g entire & $g \to 0$ as $z \to \infty$.

So for $\forall \varepsilon > 0$, we can find R_ε

s.t for $|z| > R_\varepsilon$, $|f(z)| < \varepsilon$.

Take $\varepsilon = 1$.

then for $|z| \le R_1$, as $|g|$ is cts on

this cpt set, we know $|g|$ assume

a maximum value M.

Then $|g| \le \max\{M, 2\} \xRightarrow{\text{Liouville Thm}} g$ is a

constant which is impossible. $\square$

Rk:

If some modulus of entire function has certain growth control

then it is a constant

exp: $|f| \le 1 + |z|^{\frac{1}{2}}$

$$|f'(z_0)| = \left| \frac{1}{2\pi i} \int_{B_R(z_0)} \frac{f(z)}{(z-z_0)^2} \, dz \right|$$

$$\le \frac{1}{2\pi} \int_{B_R(z_0)} \frac{|f(z)|}{R^2} \, dz$$

$$\le \frac{1+(|z_0|+R)^{\frac{1}{2}}}{R} \longrightarrow 0, \quad \text{if } R \to \infty,$$

$f(z_0) = a$ ∴ Entire f need not to be bdd to be constant, some growth control can do.